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The lift on a small sphere in wall-bounded linear shear flows

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This paper presents a closed-form solution for the inertial lift force acting on a small rigid sphere that translates parallel to a flat wall in a linear shear flow. The results provide connections between results derived by other workers for various limiting cases. An analytical form for the lift force is derived in the limit of large separations. Some new results are presented for the disturbance flow created by a small rigid sphere translating through an unbounded linear shear flow.

1. Introduction

This paper presents results for the lift force acting on a small rigid sphere in wall-bounded linear shear flows. It will be assumed that the sphere translates parallel to the wall. Such a situation might arise in the sedimentation of a particle through a vertical flow. It will also be assumed that the relevant flow Reynolds numbers are small compared to unity so that asymptotic methods may be used to derive an expression for the lift force. The results fill in gaps between the results previously published. Specifically, Saffman (1965) and McLaughlin (1991) considered the lift force acting on a small sphere in an unbounded linear shear flow. Cox & Hsu (1977) used the theory developed by Cox & Brenner (1968) to obtain analytical expressions for the migration velocity of a particle sedimenting parallel to a vertical wall. Their results are valid provided that the Reynolds number based on the distance of the particle from the wall and a characteristic flow velocity is small compared to unity. This assumption implies that the wall lies within the 'inner' region of the particle disturbance flow. Vasseur & Cox (1976) used the Cox–Brenner theory to obtain numerical results for the inertial migration velocity of a sphere sedimenting between two vertical walls.

Relatively little information is available about the situation in which the wall lies in the outer region of the disturbance flow. Vasseur & Cox (1977) removed the restriction $Re_l \ll 1$ for the case of a particle translating through a stagnant fluid next to a single planar wall or between two parallel walls. The only restriction on their analysis is that the Reynolds number based on the sphere diameter and the sedimentation velocity of the sphere should be small compared to unity.

Drew (1988) extended Saffman's analysis by including the effects of a distant wall. Drew assumed that, to zeroth order in inertial effects, the sphere moves parallel to a rigid flat wall. He further assumed that $a \ll l$, where a is the sphere radius and l is the distance between the centre of the sphere and the wall, so that the sphere may be treated as a point force acting on the fluid. Finally, he assumed that the sphere was sufficiently far from the wall that inertial effects were of the same order as viscous effects when the distance from the sphere was of order l. With these assumptions, he argued that the wall effects could be obtained by solving a fourth-order ordinary differential equation for the Fourier transform of the disturbance to the normal component of velocity due to the presence of the wall. Drew solved the ordinary differential equation by numerical means.

The work to be presented in this paper differs from that of Drew in two important respects. The first difference is that Drew, like Saffman, considered only the strong shear limit in which the two Reynolds numbers satisfy $Re_G^{\frac{1}{2}} \gg Re_s$, where

$$Re_G = Gd^2/\nu, \tag{1.1}$$

$$Re_s = v_s d/\nu. \tag{1.2}$$

In (1.1) and (1.2), d is the sphere diameter, G is the shear rate of the undisturbed flow, ν is the kinematic viscosity, and v_s is the velocity of the sphere relative to the undisturbed fluid. In this paper, no restriction is placed on the ratio $\epsilon = Re_G^{\frac{1}{2}}/Re_s$. It will only be assumed that Re_G and Re_s are small compared to unity.

The second difference between Drew's work and the present paper is that Drew solved an ordinary differential equation by numerical means to obtain the Fourier transform of the migration velocity. It will be shown that an analytical solution may be obtained in terms of one of the Airy functions. Although one must, in general, compute the Fourier integrals numerically, it will be shown that, in the limit of large distances, an analytical solution may be obtained. For small distances, the results reduce to those derived by Cox & Hsu (1977) and Vasseur & Cox (1977).

Two recent papers have discussed the effect of a distant wall on the lift force that acts on a neutrally buoyant sphere in channel flows. Schonberg & Hinch (1989) considered the lift force on a small, neutrally buoyant sphere in a plane Poiseuille flow. Drew, Schonberg & Belfort (1991) considered the lift force on a small, neutrally buoyant sphere in a laminar flow through a membrane duct. In both cases, the physics of the problem is different from the physics of the problem to be considered in this paper. A neutrally buoyant sphere produces a disturbance flow that, at large distances from the sphere, is equivalent to the disturbance created by a force dipole.

A thorough review of the literature dealing with inertial lift forces on small particles will not be attempted in this paper. Leal (1980) has reviewed the literature on inertial migration of particles at low Reynolds numbers up to 1979. There are earlier reviews by Brenner (1966), Goldsmith & Mason (1967), and Cox & Mason (1971). More recent work has been reviewed by McLaughlin (1991).

In §2, the boundary value problem to be solved will be posed and background will be provided. In §3, the asymptotic form of the disturbance flow at large distances from the sphere will be derived from the solution for the Fourier transform of the flow presented by McLaughlin (1991). It will be shown that it is possible to obtain an analytical expression for the disturbance flow in this asymptotic limit. In addition, a solution for the partial Fourier transform of the disturbance flow will be derived. The latter result will then be used in §4 to obtain results for the influence of the wall on the inertial migration velocity of (or lift force on) the sphere. Finally, the results are summarized in §5.

2. Governing equations and background

It will be assumed that a rigid sphere is located at the origin of a Cartesian coordinate system and that, in the absence of the sphere, the velocity profile is $\boldsymbol{v} = Gx\boldsymbol{e}_3$, where $\boldsymbol{e}_3$ is a unit vector in the z-direction. A planar, rigid wall is located at $x = -l$ and it is assumed that the sphere moves parallel to the wall at velocity $-v_s\boldsymbol{e}_3$. In this frame of reference, the wall moves at velocity $-Gl\boldsymbol{e}_3$. The objective of the analysis is to derive an expression for the x-component of the force acting on the particle. It is convenient to pose the problem in a frame of reference moving with the particle so that the fluid velocity field is time-independent. There are two different ways of justifying the assumption of time-independent flow. One may assume that a force equal and opposite to the lift force acts on the sphere and prevents it from migrating. An alternative is to assume that the migration velocity is very small in comparison with the sedimentation velocity so that one can treat the problem as quasi-steady. This assumption can be verified in a self-consistent manner.

The fluid surrounding the sphere is incompressible and Newtonian. When written in terms of the disturbance velocity field created by the sphere, $\boldsymbol{v}$, the Navier–Stokes equation takes the form

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + (v_s + Gx)\,\partial\boldsymbol{v}/\partial z + Gv_1\,\boldsymbol{e}_3 = -\boldsymbol{\nabla}p/\rho + \nu\nabla^2\boldsymbol{v}. \tag{2.1}$$

In (2.1), the symbol p denotes the pressure in the fluid, ρ denotes the fluid density, and ν denotes the kinematic viscosity of the fluid. The boundary conditions on $\boldsymbol{v}$ are that it must vanish at infinite distance from the sphere and it must be consistent with rigid no-slip boundary conditions on the surface of the sphere.

Even though Re_G and Re_s are small compared to unity, at sufficiently large distances from the sphere, inertial effects are comparable in magnitude to viscous effects. In this outer region, the Navier–Stokes equation may be approximated by

$$(v_s + Gx)\,\partial\boldsymbol{v}/\partial z + Gv_1\,\boldsymbol{e}_3 = -\boldsymbol{\nabla}p/\rho + \nu\nabla^2 v - (\boldsymbol{F}/\rho)\,\delta(\boldsymbol{r}). \tag{2.2}$$

In (2.2), $\boldsymbol{r}$ denotes the position vector of a point in the fluid and $\boldsymbol{F}$ denotes the force exerted by the fluid on the particle, to zeroth order in inertial effects, $\boldsymbol{F} = 6\pi\mu a v_s\,\boldsymbol{e}_3$, where a is the sphere radius and μ is the dynamic viscosity of the fluid.

The disturbance flow velocity field is assumed to be incompressible:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0. \tag{2.3}$$

McLaughlin (1991) considered the case of a sphere translating through an unbounded fluid. In this case, one assumes that the disturbance flow vanishes at large distances from the particle,

$$\boldsymbol{v} = 0, \quad \boldsymbol{r} = \infty. \tag{2.4}$$

McLaughlin (1991) showed that the migration velocity in the x-direction, v_m, may be expressed as follows:

$$v_m = (3/2\pi^2)\,av_s(G/\nu)^{\frac{1}{2}}J, \tag{2.5}$$

where

$$J = \int_0^{2\pi}\int_0^1\int_0^\infty\left[\zeta\{s^2 - 2s^2(1-s^2)\cos^2\phi - \zeta s^3(1-s^2)^{\frac{1}{2}}\cos\phi\}\left(\frac{\pi^{\frac{1}{2}}}{4A^3}\right)\left(1-\frac{B^2}{2A^2}\right)\right.$$
$$\left. -\left(\frac{\pi^{\frac{1}{2}}B}{4\epsilon A^3}\right)s^2(1-s^2)^{\frac{1}{2}}\cos\phi\right]\mathrm{e}^{-B^2/4A^2}\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}\phi. \tag{2.6}$$

In (2.6),

$$A^2 = \tfrac{1}{3}s^2\zeta^3 + s(1-s^2)^{\frac{1}{2}}\cos\phi\,\zeta^2 + \zeta, \tag{2.7}$$

$$B = \zeta s/\epsilon, \tag{2.8}$$

and ϵ is defined by

$$\epsilon = Re_G^{\frac{1}{2}}/Re_s. \tag{2.9}$$

In the above equations, $s = \cos\theta$ and ϕ and θ denote the angular coordinates of a spherical system in Fourier space.

For large ϵ, with an error of order $1/\epsilon^4$, J may be approximated by

$$J = 2.255 - 0.6463/\epsilon^2. \tag{2.10}$$

For $\epsilon = 1$, the error involved in using the asymptotic formula to compute J is 3.4% and the error involved in using the asymptotic formula to compute the difference between J and Saffman's value for J (2.255) is 7.7%.

The leading behaviour of J in the limit $\epsilon \ll 1$ is

$$J = -32\pi^2\epsilon^5 \ln(1/\epsilon^2). \tag{2.11}$$

The lowest-order corrections to (2.11) are of order ϵ^5. At intermediate values of ϵ, J must be evaluated by numerical integration, and the results are tabulated by McLaughlin (1991).

The lift force is related to the migration velocity, v_m, by

$$f_t = 6\pi\mu a v_m. \tag{2.12}$$

Some insight into the origin of the inertial lift force may be obtained by considering the characteristic scales of the problem and the relative importance of the convective and viscous terms in (2.1). Two lengths of interest are the Stokes length, $L_s = \nu/v_s$, and the Saffman length, $L_G = (\nu/G)^{\frac{1}{2}}$. For $\epsilon \gg 1$, $L_s \gg L_G$ and, for $\epsilon \ll 1$, $L_s \ll L_G$. For small values of Re_G and Re_s, the viscous term in (2.1) is small compared to the convective terms provided that the distance from the centre of the sphere, r, is small compared to both L_G and L_s.

If $\epsilon \gg 1$ (the case considered by Saffman), inertia will become significant when $r \sim L_G$. For distances of this order, the terms $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ and $v_s\,\partial\boldsymbol{v}/\partial z$ may be neglected in comparison with the terms involving G. In fact, the terms involving G remain dominant at distances that are large compared with L_G. As shown by Saffman, the lift force is caused by a transverse component of the disturbance flow that originates at distances of order L_G. The form of the lift force can be guessed on the basis of dimensional analysis guided by this intuitive notion. It is plausible that the inertial migration velocity should be proportional to v_s and that it should involve L_G. Thus, one might guess that the inertial migration velocity should be proportional to $v_s a/L_G = v_s a(G/\nu)^{\frac{1}{2}}$. To obtain the lift force, one uses (2.12).

When $\epsilon \ll 1$, the Stokes length is small compared to the Saffman length. For distances from the sphere, r, satisfying $r \sim L_s$, the term involving v_s in (2.1) is comparable to the viscous term. In addition, the term involving v_s is larger than the other convective terms. Thus, for distances satisfying $r \ll L_G/\epsilon$, the disturbance flow should be well approximated by axisymmetric Oseen flow. Since there is no lift in an axisymmetric flow, it is plausible that the lift force should be very small compared to the Saffman lift force, which ignores the convective term involving v_s. The terms involving G in (2.1) become large compared to the terms involving v_s for $r \gg L_G/\epsilon$. However, the disturbance flow has decayed to very small values at such large distances. The above estimates do not apply in the Oseen wake. Within the Oseen wake, the terms involving G in (2.1) are comparable in magnitude to the terms involving v_s at points satisfying $r \sim \epsilon^3 L_G$. However, inertial effects will only become important for $r \sim L_G$ within the wake.

3. Disturbance flow in an unbounded fluid

In this section, the solution of (2.2) for the case of an unbounded fluid will be obtained. The results to be obtained here will be used in §4 to obtain the effect of a distant wall on the inertial lift force. For this purpose, it is convenient to introduce the Fourier transforms of the velocity field and the pressure field:

$$\boldsymbol{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{u}\, \mathrm{e}^{\mathrm{i}(k_1 x + k_2 y + k_3 z)}\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3, \tag{3.1}$$

and

$$p = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi\, \mathrm{e}^{\mathrm{i}(k_1 x + k_2 y + k_3 z)}\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3. \tag{3.2}$$

By substituting the Fourier transforms in (3.1) and (3.2) into (2.2), one obtains an ordinary differential equation for $\boldsymbol{u}$. McLaughlin (1991) has given the solution of the differential equation for u_1:

$$u_1 = \frac{3}{4\pi^2}\frac{\nu a v_s k_3}{G k^2}\int_0^{\infty} \mathrm{e}^{(\psi'-\psi)}\,(\zeta k_3 + k_1)\,\mathrm{d}\zeta, \tag{3.3}$$

where

$$\psi' - \psi = -\frac{\nu}{3G}k_3^2\zeta^3 - \frac{\nu}{G}k_1 k_3 \zeta^2 - \frac{\nu}{G}k^2\zeta - \frac{\mathrm{i}v_s}{G}k_3\zeta. \tag{3.4}$$

The expression for u_1 in (3.3) is valid regardless of the sign of Gk_3.

To investigate the behaviour of the disturbance flow at large distances from the sphere, it is convenient to introduce a partial Fourier transform, $\boldsymbol{\Gamma}$:

$$\boldsymbol{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{\Gamma}\, \mathrm{e}^{\mathrm{i}(k_2 y + k_3 z)}\, \mathrm{d}k_2\, \mathrm{d}k_3. \tag{3.5}$$

The expression for u_1 in (3.3) maybe used to derive an expression for Γ_1. First, an asymptotic form for u_1 will be derived. Let us consider a point at distance r from the sphere that is large compared to L_G. For $r \gg L_G$, $k \sim 1/r \ll 1/L_G$, and the dominant contributions to u_1 come from values of ζ of order $(r/L_G)^{\frac{2}{3}}$. This result maybe obtained by assuming that the first term of (3.4) is dominant. It is also assumed that ϵ is fixed. It follows that u_1 is given by the following asymptotic approximation:

$$u_1 = \frac{3}{4\pi^2}\frac{\nu a v_s}{G}\frac{q_3^{\frac{2}{3}}}{q^2}J_1, \tag{3.6}$$

where the dimensionless wavevector $\boldsymbol{q} = (G/\nu)^{-\frac{1}{2}}\boldsymbol{k}$ has been introduced and the dimensionless integral J_1 is defined by

$$J_1 = \int_0^{\infty} g\, \mathrm{e}^{-g^3/3}\, \mathrm{d}g. \tag{3.7}$$

It may be shown that $J_1 = 3^{-\frac{1}{3}}\Gamma(\frac{2}{3})$. Davis (1965) gives tables of the Γ-function. To four decimals, the value of J_1 is 0.9389. The value of Γ_1 is obtained by performing the one-dimensional Fourier integral of (3.6):

$$\Gamma_1 = (3/4\pi^2)\,(\nu/G)^{\frac{1}{2}} a v_s J_1 q_3^{\frac{2}{3}} I, \tag{3.8}$$

where
$$I = \int_{-\infty}^{\infty} \frac{e^{iq_1 x_*}}{q^2} dq_1. \tag{3.9}$$

The dimensionless coordinate x_* is equal to x/L_G. It may be shown, using contour integration, that
$$I = (\pi/p)\, e^{-p|x_*|}, \tag{3.10}$$

where $p^2 = q_2^2 + q_3^2$. Finally, when (3.8) and (3.10) are used with (3.5), one obtains the following expression for the x-component of the disturbance flow along the line $y = z = 0$:
$$v_1 = (3J_1/4\pi)\, av_s(G/\nu)^{\frac{1}{2}} J_2, \tag{3.11}$$

where
$$J_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{q_3^{\frac{2}{3}}}{p} e^{-p|x_*|}\, dq_2\, dq_3. \tag{3.12}$$

It may be shown that $J_2 = J_3 J_4/|x_*|^{\frac{5}{3}}$, where
$$J_3 = 4\Gamma(\tfrac{5}{3}) \tag{3.13}$$

and
$$J_4 = 2^{-\frac{1}{3}}(\Gamma(\tfrac{5}{6}))^2/\Gamma(\tfrac{5}{3}). \tag{3.14}$$

To four decimals, the value of J_2 is $4.044/|x_*|^{\frac{5}{3}}$. Thus, the x-component of the disturbance flow at $y = z = 0$ is
$$v_1 = 0.9064 av_s(G/\nu)^{\frac{1}{2}}/(|x_*|^{\frac{5}{3}}). \tag{3.15}$$

For $v_s > 0$, the x-component of the velocity for $y = z = 0$ points in the direction of increasing fluid velocity. It is assumed that $G > 0$ in writing the above equations.

The above results are valid for any value of ϵ. For smaller values of $|x_*|$, the x-component of velocity depends on the value of ϵ as well as on the value of x_*. The value of v_1 was determined by numerical evaluation of the Fourier integral in (3.1). The IMSL routine DQAND was used for this purpose. Further details may be found in McLaughlin (1991). In all cases, $y = z = 0$. The computed velocity field is valid only for values of r that satisfy $r \gg a$ because of the Oseen-like approximation for the nonlinear term of the Navier–Stokes equation. In addition, the point force approximation breaks down for $r = O(a)$. Thus, the approximation breaks down for $x \sim L_s Re_s$ or $x \sim L_G Re_G^{\frac{1}{2}}$.

In computing the Fourier integral, the procedures described by McLaughlin (1991) may be used to show that
$$v_1 = (3/2\pi^2)\, av_s\, (G/\nu)^{\frac{1}{2}} J, \tag{3.16}$$

where J has the same form as in (2.6) except that
$$B = \zeta s/\epsilon + x_*(1-s^2)^{\frac{1}{2}} \cos\phi. \tag{3.17}$$

In the above equations, s and ϕ have the same meaning as in (2.6). Figure 1 illustrates the behaviour of J as a function of x_* for $\epsilon = \infty$ (the Saffman limit), $\epsilon = 1$, and $\epsilon = 0.2$. For $\epsilon = \infty$, v_1 is an even function of x for $y = z = 0$. For $\epsilon = 1$, figure 1 reveals a strong asymmetry, but v_1 is still positive for all values of x. For $\epsilon = 0.2$, v_1 is nearly antisymmetric. For finite values of ϵ, there is a jump discontinuity at the origin. The discontinuity is not physical; it is caused by the failure of the Oseen approximation at small values of x_*.

The behaviour of J and v_1 for small values of ϵ is of particular interest since

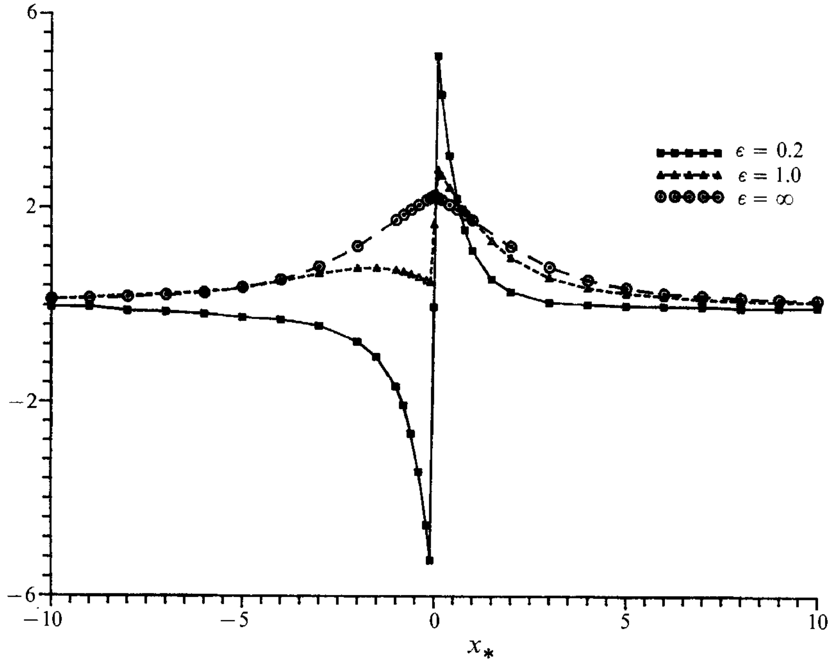

FIGURE 1. The values of J in (3.16) obtained by numerical integration for $\epsilon = 0.2$, 1.0, and ∞.

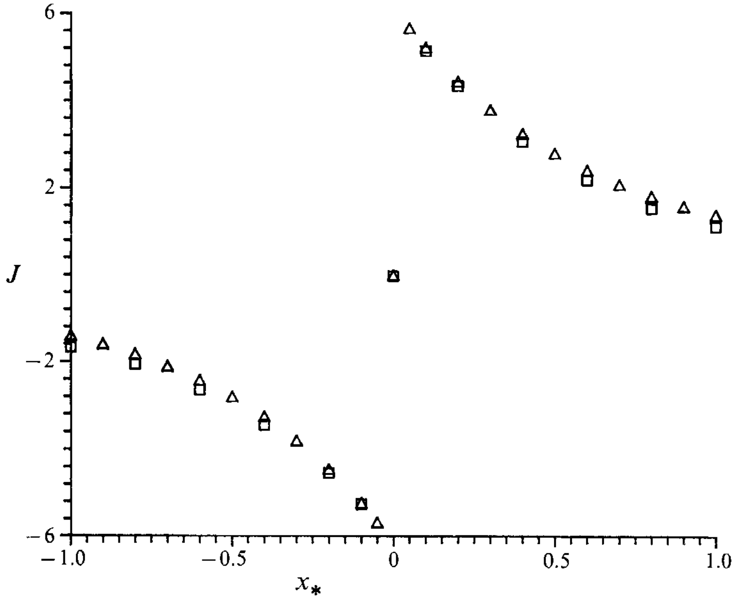

FIGURE 2. For $\epsilon = 0.2$, the values of J in (3.16) obtained by numerical integration (□) are compared with the values predicted by the Oseen approximation (△).

McLaughlin's (1991) results indicate that the inertial migration velocity changes sign for $\epsilon \approx 0.22$. Figure 2 shows that v_1 is well approximated by the Oseen disturbance flow for $\epsilon = 0.2$. In this case, for $x_* \leqslant 5$, the shear terms in the Navier–Stokes equation are small compared with the convective terms involving v_s.

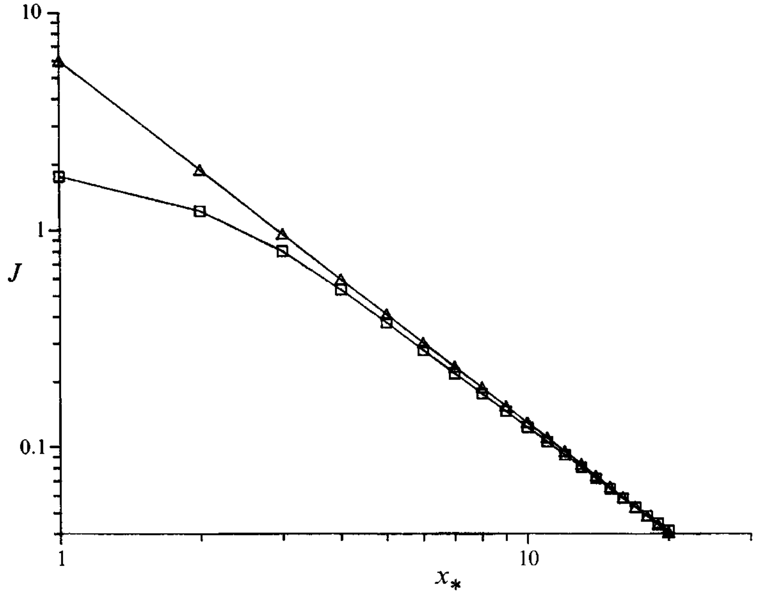

FIGURE 3. For $\epsilon = \infty$, the values of J in (3.16) obtained by numerical integration (□) are compared with the asymptotic result in (3.15) (△).

Finally, figure 3, which is for $\epsilon = \infty$, shows that the numerical results for v_1 are consistent with the asymptotic result derived earlier for $x_* \gg 1$.

To incorporate wall effects, a solution for the partial Fourier transform, Γ_1, is useful. If (3.5) is substituted into (2.2), and one eliminates the pressure by using the continuity equation, one obtains the following fourth-order ordinary equation for the partial Fourier transform:

$$\left[\mathrm{i}q_3\left(\frac{1}{\epsilon}+x_*\right)-\left(\frac{\partial^2}{\partial x_*^2}-p^2\right)\right]\left(\frac{\partial^2}{\partial x_*^2}-p^2\right)\Gamma_1^* = \frac{3\mathrm{i}q_3}{\pi Re_G^{\frac{1}{2}}}\frac{\mathrm{d}\delta(x_*)}{\mathrm{d}x_*}. \tag{3.18}$$

In (3.18), the dependent variable is the dimensionless partial Fourier transform, $\Gamma_1^* = \Gamma_1/a^2 v_s$. To find a solution of the equation, it is helpful to write the left-hand side of the equation in terms of two linear operators:

$$\mathrm{L}_1\mathrm{L}_2\Gamma_1^* = \frac{3\mathrm{i}q_3}{\pi Re_G^{\frac{1}{2}}}\frac{\mathrm{d}\delta(x_*)}{\mathrm{d}x_*}. \tag{3.19}$$

The linear operators are defined by

$$\mathrm{L}_1 = \partial^2/\partial(\gamma x_* + \beta^2/\gamma^2)^2 - (\gamma x_* + \beta^2/\gamma^2) \tag{3.20}$$

and

$$\mathrm{L}_2 = \partial^2/\partial x_*^2 - p^2. \tag{3.21}$$

The quantity γ is defined to be $(\mathrm{i}q_3)^{\frac{1}{3}}$, where the branch is chosen so that the angle measured from the real axis is 30° in magnitude. The quantity β is defined by

$$\beta^2 = p^2 + \mathrm{i}q_3/\epsilon. \tag{3.22}$$

As a first step, one solves the following equation:

$$\mathrm{L}_1 f = \frac{3\mathrm{i}q_3}{\pi Re_G^{\frac{1}{2}}}\frac{\mathrm{d}\delta(x_*)}{\mathrm{d}x_*}. \tag{3.23}$$

A solution of the homogeneous problem that decays as x_* goes to infinity is the Airy function $\mathrm{Ai}\,(\gamma x_* + \beta^2/\gamma^2)$. Another, independent, solution that decays as x_* goes to negative infinity is $\mathrm{Ai}\,(\omega^2[\gamma x_* + \beta^2/\gamma^2])$. The quantity ω is defined to be $\mathrm{e}^{-2\pi\mathrm{i}/3}$. Bender & Orszag (1978) and Antosiewicz (1965) discuss the properties of the Airy functions and their asymptotic representations. The notation and calculational procedures in this paper may be found in those references.

One may find the solution of (3.23) by the method of 'variation of parameters'. For positive values of x_*,

$$f = -\frac{3\mathrm{i}q_3}{\pi c Re_G^{\frac{1}{2}}\gamma}\mathrm{Ai}\,(\gamma x_* + \beta^2/\gamma^2)\left.\frac{\partial\,\mathrm{Ai}\,(\omega^2[\gamma s + \beta^2/\gamma^2])}{\partial s}\right|_{s=0}. \tag{3.24}$$

For negative values of x_*,

$$f = -\frac{3\mathrm{i}q_3}{\pi c Re_G^{\frac{1}{2}}\gamma}\mathrm{Ai}\,(\omega^2[\gamma x_* + \beta^2/\gamma^2])\left.\frac{\partial\,\mathrm{Ai}\,(\gamma s + \beta^2/\gamma^2)}{\partial s}\right|_{s=0}. \tag{3.25}$$

In (3.24) and (3.25), the constant c is given by

$$c = \mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0)\,[-1+\omega^2]. \tag{3.26}$$

Finally, one must solve the following equation:

$$\mathrm{L}_2\,\Gamma_1^* = f. \tag{3.27}$$

The boundary conditions on Γ_1^* are that it must vanish at infinity. One may find the solution by variation of parameters:

$$\Gamma_1^* = \frac{\mathrm{e}^{px_*}}{2p}\int_{\infty}^{x_*}\mathrm{e}^{-ps}f(s)\,\mathrm{d}s - \frac{\mathrm{e}^{-px_*}}{2p}\int_{-\infty}^{x_*}\mathrm{e}^{ps}f(s)\,\mathrm{d}s. \tag{3.28}$$

4. Wall effects

When a wall is present at $x = -l$, the normal component of velocity and its first derivative with respect to the normal component must vanish at $x = -l$. The latter condition follows from the continuity equation and the fact that the other two components of velocity must vanish on the wall. In terms of the Fourier transform, Γ_1, these conditions are

$$\Gamma_1 = 0, \quad x = -l \tag{4.1}$$

$$\mathrm{d}\Gamma_1/\mathrm{d}x = 0, \quad x = -l. \tag{4.2}$$

Since (2.2) is linear, one may appeal to the principle of superposition to write Γ_1 in terms of the value that it would have in an unbounded fluid and the disturbance created by the wall:

$$\Gamma_1 = \Gamma_1^{\mathrm{u}} + \Gamma_1^{\mathrm{w}}, \tag{4.3}$$

where the superscripts u and w denote the unbounded solution and the wall flow, respectively.

The wall contribution to the partial Fourier transform satisfies the homogeneous version of (3.19):

$$\mathrm{L}_1\,\mathrm{L}_2\,\Gamma_1^{\mathrm{w}} = 0. \tag{4.4}$$

One may obtain the solution of (4.4) by employing the same procedure as in §3. First, one seeks a solution of

$$\mathrm{L}_1 f^{\mathrm{w}} = 0 \tag{4.5}$$

that vanishes at $x_* = \infty$. The solution is

$$f^{\mathrm{w}} = c_1 \operatorname{Ai}(\gamma x_* + \beta^2/\gamma^2). \tag{4.6}$$

The solution for $\Gamma_1^{\mathrm{w}*}$ $(= \Gamma_1/(a^2 v_{\mathrm{s}}))$ is

$$\Gamma_1^{\mathrm{w}*} = \frac{\mathrm{e}^{px_*}}{2p}\int_{\infty}^{x_*} \mathrm{e}^{-ps} f^{\mathrm{w}}(s)\,\mathrm{d}s - \frac{\mathrm{e}^{-px_*}}{2p}\int_{-l_*}^{x_*} \mathrm{e}^{ps} f^{\mathrm{w}}(s)\,\mathrm{d}s + c_2\,\mathrm{e}^{-p(x_*+l_*)}. \tag{4.7}$$

One may determine the constants c_1 and c_2 from the boundary conditions in (4.1) and (4.2) with the expressions for $\Gamma_1^{\mathrm{u}*}$ in (3.28):

$$c_1 = (p\Gamma_1^{\mathrm{u}*}(-l_*) + \mathrm{d}\Gamma_1^{\mathrm{u}*}/\mathrm{d}x_*|_{x_*=-l_*})/I_1, \tag{4.8}$$

$$c_2 = \tfrac{1}{2}(-\Gamma_1^{\mathrm{u}*}(-l_*) + (1/p)\,\mathrm{d}\Gamma_1^{\mathrm{u}*}/\mathrm{d}x_*|_{x_*=-l_*}). \tag{4.9}$$

In (4.8), the quantity I_1 is defined as

$$I_1 = \int_{-l_*}^{\infty} \mathrm{e}^{-p(s+l_*)} \operatorname{Ai}\left(\gamma s + \frac{\beta^2}{\gamma^2}\right)\mathrm{d}s. \tag{4.10}$$

Finally, by performing the integrals in (3.5) at $x = y = z = 0$, one obtains the inertial migration velocity of the sphere (see Saffman 1965 for a justification of this procedure):

$$v_{\mathrm{m}} = v_{\mathrm{m}}^{\mathrm{u}} + v_{\mathrm{m}}^{\mathrm{w}}. \tag{4.11}$$

In (4.11), $v_{\mathrm{m}}^{\mathrm{u}}$ denotes the inertial migration velocity of the sphere in an unbounded fluid. The value of $v_{\mathrm{m}}^{\mathrm{u}}$ may be determined from (2.5) and (2.6).

For large values of l_*, one may obtain an analytical expression for the wall contribution to the migration velocity. By substituting (3.8) into (4.8) and (4.9), one finds that $c_2 = 0$ and $c_1 = 2p\Gamma_1^{\mathrm{u}}/I_1$. Furthermore, for large values of l_*, the first integral in (4.7) is negligible compared to the second integral because of the behaviour of Ai for arguments that are large in magnitude. Thus, the partial Fourier transform of the wall contribution to the inertial migration velocity is given by

$$\Gamma_1^{\mathrm{w}*} = -\frac{3}{2\pi}\frac{0.9389\,q_3^{\frac{2}{3}}}{Re_G^{\frac{1}{2}}\;p}\,\mathrm{e}^{-pl_*}\frac{I_2}{I_1}, \tag{4.12}$$

where

$$I_2 = \int_{-l_*}^{0} \mathrm{e}^{ps} \operatorname{Ai}\left(\gamma s + \frac{\beta^2}{\gamma^2}\right)\mathrm{d}s. \tag{4.13}$$

In the limit of large l_*, only values of p that are order $1/l_*$ are significant. In this limit, I_2/I_1 approaches e^{-pl_*}. When the forward transform is evaluated, the result is

$$v_{\mathrm{m}}^{\mathrm{w}} = -0.2855 a v_{\mathrm{s}} (G/\nu)^{\frac{1}{2}}/l_*^{\frac{5}{3}}. \tag{4.14}$$

When G is negative, G must be replaced in (4.14) by its absolute magnitude and the sign of the expression on the right-hand side of the equation must be changed.

Vasseur & Cox (1977) pointed out that one may appeal to boundary-layer theory to obtain the effects of a distant wall to leading order. Although Re_G and Re_{s} are small compared to unity, at large distances from the sphere the effects of inertia dominate viscous effects except in the Oseen wake. Inertia will become dominant at distances $r \gg \min(L_G, L_{\mathrm{s}})$ except in the Oseen wake. Thus, if the wall is at a distance l satisfying $l \gg \min(L_G, L_{\mathrm{s}})$, it may be assumed that, near the wall, the disturbance flow created by the sphere is inviscid. Very close to the wall, viscous effects will be important since both (4.1) and (4.2) must be satisfied on the wall. However, outside a boundary layer of thickness δ, the wall disturbance flow may be treated as inviscid. The boundary-layer thickness may be estimated as $\delta \sim (\nu l / v_s)^{\frac{1}{2}}$. Thus, viscous effects will be eligible outside a thin region provided that $\delta \ll l$, or $L_s \ll l$.

Based on the above argument, one may obtain the result in (4.14) with the method of images. The wall-induced inertial migration velocity is obtained by replacing x_* by $2l_*$ in (3.15) and reversing the sign of the result.

The sign of v_m^w in (4.14) indicates that the wall exerts an attractive force on the sphere when the product $v_s G$ is positive and a repulsive force when $v_s G < 0$. This appears to be inconsistent with Drew's (1988) numerical results for $\epsilon = \infty$. Drew states that, when $v_s G$ is positive (so that the particle experiences a lift force that points away from the wall), the wall disturbance tends to increase the magnitude of the lift force for $l_* \gg 1$. However, the analysis of this paper indicates that the reverse is true.

In the weak shear limit, $\epsilon \ll 1$, the disturbance flow is well approximated by the Oseen differential equation for distances r satisfying $a \ll r \ll L_G/\epsilon$ as was shown in §3. Vasseur & Cox (1977) have shown that, in this case, the sign of v_m^w is always positive. The physical mechanism is that, as the sphere translates parallel to the wall, it displaces fluid laterally and the wall creates a counterflow that pushes the sphere away from the wall. For $\epsilon \ll 1$, the asymptotic result in (4.14) applies only for $l_* = O(1/\epsilon^3)$. This may be seen by considering the argument leading to (3.15). If k is $O(1/l)$, the cubic term in (3.4) will be order unity if $\zeta = O(l_*^{\frac{2}{3}})$. The first term will dominate the other terms provided that $\zeta^3/l_*^2 \gg \zeta/(\epsilon l_*)$. Thus, the power-law behaviour will be valid only if $l_* \gg 1/\epsilon^3$.

For values of l_* that are order unity, one must evaluate the integrals in (4.7) and (3.5) to determine the effect of the wall on the inertial migration velocity. The Airy function, Ai, must be evaluated for complex arguments in order to compute Γ_1^u as given by (3.28), as well as the integrals in (4.7) and (4.10). For values of the argument smaller in magnitude than 4, the Taylor series expansion of Ai was used to obtain its value. The number of terms in the series was chosen to be $(10|z|^3)^{\frac{1}{2}}$, where $|z|$ is the magnitude of the complex argument. For values of the argument larger in magnitude than 4, the first three terms in the asymptotic series were used to obtain Ai. A convenient summary of the relevant expansions may be found in Bender & Orszag (1978). Shibata & Mei (1990) used a very similar procedure to obtain the values of the Airy function for complex arguments. One minor difference is that Shibata & Mei used the asymptotic series to evaluate the Airy functions when the magnitude is greater than 5 instead of 4. This procedure is slightly more accurate, but also more time-consuming.

The trapezoid rule was used to compute the relevant integrals. To compute the integrals in (3.5), polar coordinates were used. Symmetries were exploited to reduce the computation of the integral over the polar angle to an integral from 0 to $\frac{1}{2}\pi$. In the angular integral, 11 grid points were used. The integral over p was broken into four shells: 0 to 1, 1 to 10, 10 to 100, and 100 to 1000. Each shell contained 400 grid points. For large values of l_*, only the contribution from the first shell was significant. A similar procedure was followed for the s-integrals except that only three shells were used in these calculations. Each s shell contained 160 grid points.

The largest errors in J were associated with the s-integrations. The errors were largest for small values of ϵ since more grid points were needed to resolve phenomena on the Stokes scale, $L_s = \epsilon L_G$. For $\epsilon = 0.2$, the largest error in J is 0.066 in magnitude, or 2.1 %, and it occurs for $l_* = 0.1$. The largest percentage error is 5.2 % and it occurs

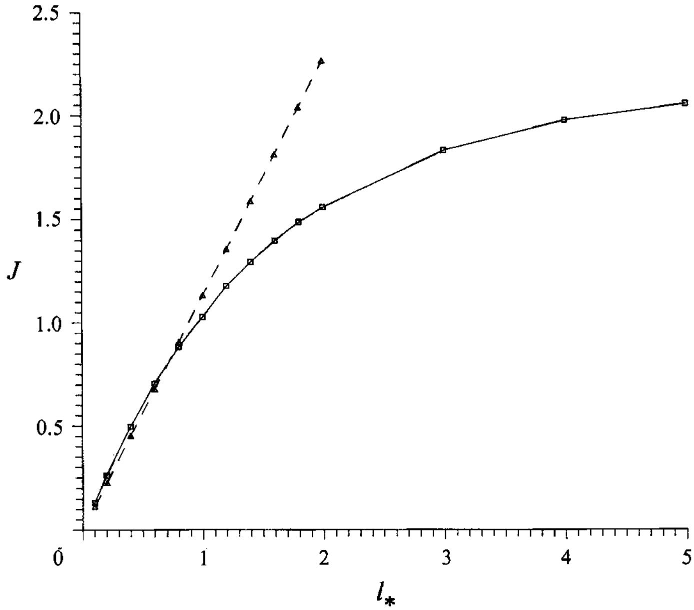

FIGURE 4. The values of J in (4.15) obtained by numerical integration for $\epsilon = \infty$ (□) are compared with the Cox–Hsu theory (△).

for $l_* = 1.2$. For $\epsilon = 1.0$, the largest error is 0.018 in magnitude, or 2.5%, and it occurs for $l_* = 0.1$. The above error is also the largest percentage error. Finally, for $\epsilon = \infty$, the largest error is 0.021 in magnitude, or 2.1%, and it occurs for $l_* = 1.2$. The largest percentage error is 3.4% and it occurs for $l_* = 0.1$.

To present results for the inertial migration velocity, the same non-dimensional quantity, J, will be used in order to facilitate comparisons with the results for the migration velocity in an unbounded fluid and the disturbance flow in an unbounded fluid:

$$J = \frac{2\pi^2}{3}\left(\frac{\nu}{G}\right)^{\frac{1}{2}} \frac{v_m}{a v_s} \tag{4.15}$$

and

$$J = J^u + J^w, \tag{4.16}$$

where the superscripts u and w denote the values in an unbounded fluid (given in (2.6)) and the wall contribution, respectively.

Figure 4 shows the values of J versus l_* for $\epsilon = \infty$. The predictions of the Cox–Hsu theory for small values of l_* are also plotted in the figure.

Cox & Hsu (1977) considered the problem of a small sphere sedimenting in a vertical parabolic flow next to a vertical rigid wall. They considered three cases: a non-neutrally buoyant sphere in a strong shear flow; a non-neutrally buoyant sphere in a weak shear flow; and a neutrally buoyant sphere. The first two cases are relevant for comparison with the cases considered in the present paper. The Cox–Hsu theory provides the following expression for the inertial migration of a non-neutrally buoyant sphere in a vertical linear shear flow:

$$v_m = \tfrac{3}{32}\frac{a v_s^2}{\nu} + \tfrac{11}{64}\frac{G a v_s l}{\nu}. \tag{4.17}$$

The result in (4.17) is valid provided that $a \ll l \ll \min(L_G, L_s)$. In other words, the wall is assumed to lie within an 'inner' region where inertial effects are a small perturbation of the Stokes equation.

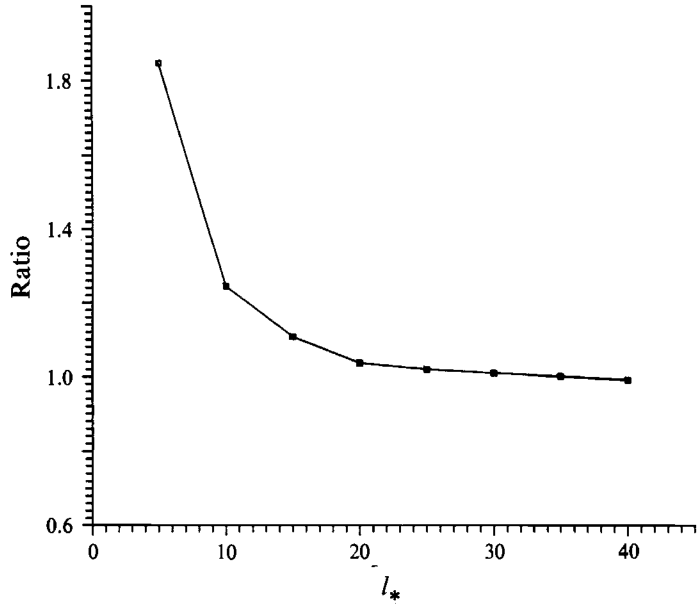

FIGURE 5. For $\epsilon = \infty$, the ratio of $|J^w|$ in (4.16) obtained by numerical integration to the asymptotic prediction in (4.14) is shown.

If the expression for v_m (4.17) is written in terms of the 'outer' coordinate, l_*, one obtains the following expression for J:

$$J = \frac{\pi^2}{16}\left(\frac{1}{\epsilon}+\frac{11}{6}l_*\right). \tag{4.18}$$

In the strong shear limit ($\epsilon \gg 1$), the second term dominates.

The results derived in this paper must reduce to the Cox–Hsu result for sufficiently small values of l_*. The Cox–Hsu theory also treats the particle as a point force acting on the fluid (for non-neutrally buoyant particles.) Furthermore, the nonlinear term in the Navier–Stokes equation may be approximated by the form in (2.2) to leading order for $r \gg a$, where r is the distance from the centre of the sphere. The primary difference is that, within its domain of validity, the Cox–Hsu theory treats the nonlinear term as small compared to the viscous term so that ordinary perturbation methods may be used to obtain the leading-order result. Cox & Hsu divide the flow field into inner and outer regions on the basis of whether the distance from the sphere is comparable to the sphere's radius (the inner region) or whether the distance from the sphere is comparable to the distance of the sphere from the wall. In both regions, the inertial terms are small compared to the viscous terms. In the outer region, the sphere is treated as a point force to leading order. Cox & Hsu show that the leading contribution to the lift force comes from the outer region. All terms in (2.1) are treated in their analysis. However, the term $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ contributes at a higher order in the ratio a/l than the other terms. Thus, to leading order, the nonlinear term may be treated as in (2.2) and the results of this paper must reduce to the Cox–Hsu theory for sufficiently small values of l.

In figure 4, it may be seen that, for $\epsilon = \infty$, J converges to the Cox–Hsu theory for small values of l_*. In figure 5, the ratio of the computed value of J^w to the power-law value is plotted versus l_* for $\epsilon = \infty$. For values of $\epsilon > 1$, the power-law formula gives estimates for J^w that are accurate to within 25% for $l_* > 10$.

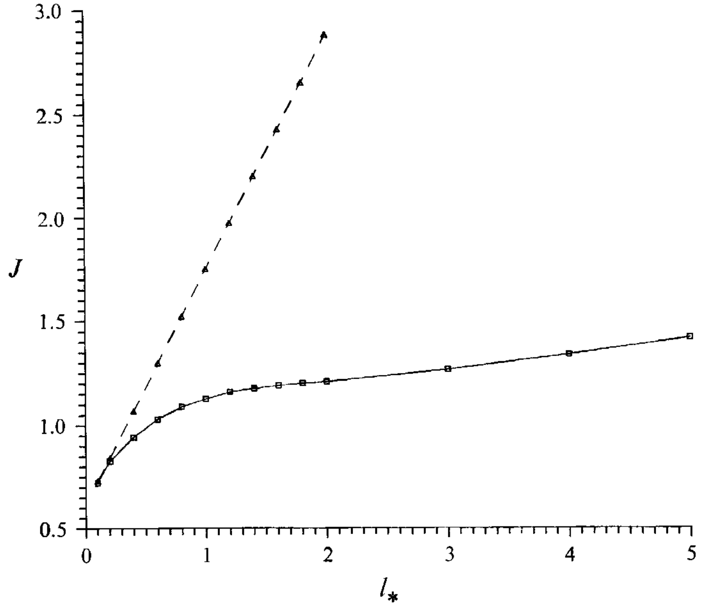

FIGURE 6. The values of J in (4.15) obtained by numerical integration for $\epsilon = 1$ (□) are compared with the Cox–Hsu theory (△).

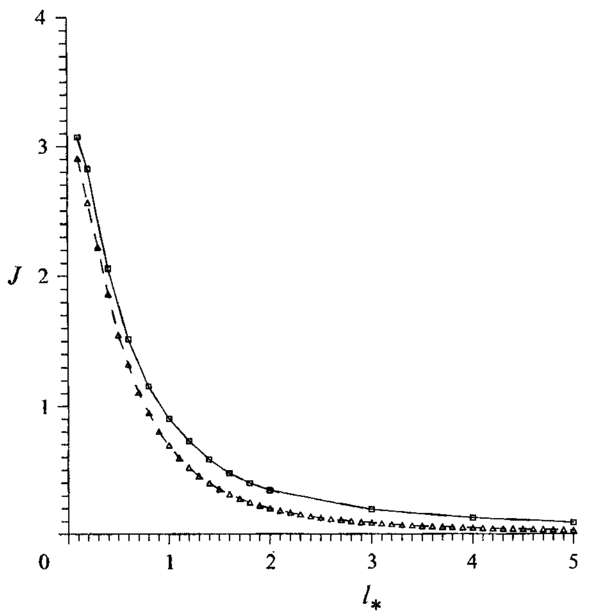

FIGURE 7. The values of J in (4.15) obtained by numerical integration for $\epsilon = 0.2$ (□) are compared with the Vasseur–Cox theory (△).

In figure 6, the values of J are plotted versus l_* for $\epsilon = 1$. The computed results agree well with the Cox–Hsu theory for small values of l_*.

For small values of ϵ, the Cox–Hsu theory is valid only for $l_* \ll \epsilon$. Within its small region of validity, the Cox–Hsu theory predicts that the shear contribution to the lift is unimportant. For $Re_G^{\frac{1}{2}} \ll l_* \ll 1/\epsilon$, the Vasseur–Cox (1977) theory should provide

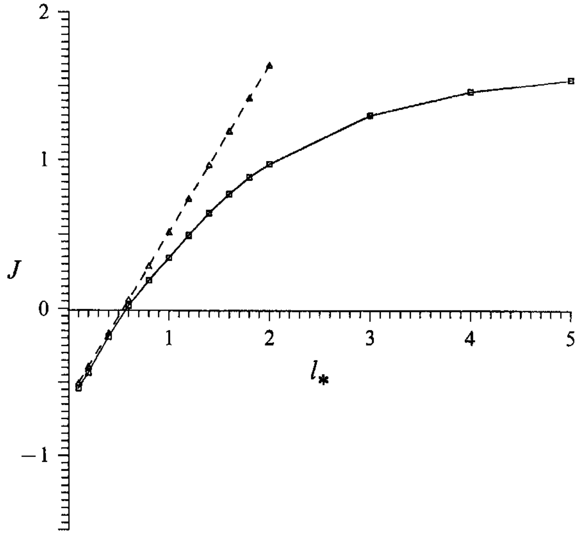

FIGURE 8. The values of J in (4.15) obtained by numerical integration for $\epsilon = -1$ (□) are compared with the Cox–Hsu theory (△).

l_*	$\epsilon = 0.2$	0.4	0.6	0.8	1.0	1.5	2.0	∞
0.1	3.07	1.65	1.14	0.881	0.720	0.505	0.409	0.143
0.2	2.82	1.69	1.23	0.982	0.826	0.615	0.521	0.255
0.4	2.06	1.56	1.25	1.07	0.943	0.766	0.686	0.455
0.6	1.52	1.42	1.25	1.12	1.03	0.891	0.827	0.631
0.8	1.16	1.30	1.23	1.15	1.09	0.983	0.934	0.771
1.0	0.903	1.19	1.20	1.17	1.13	1.05	1.01	0.886
1.2	0.727	1.08	1.17	1.18	1.16	1.12	1.10	1.01
1.4	0.580	0.977	1.12	1.17	1.18	1.17	1.17	1.12
1.6	0.475	0.889	1.08	1.16	1.19	1.21	1.23	1.22
1.8	0.398	0.816	1.04	1.15	1.20	1.25	1.52	1.30
2.0	0.342	0.766	1.01	1.14	1.21	1.28	1.69	1.37
3.0	0.192	0.572	0.908	1.13	1.27	1.44	1.52	1.69
4.0	0.126	0.463	0.857	1.15	1.34	1.58	1.69	1.89
5.0	0.090	0.396	0.848	1.19	1.42	1.70	1.82	2.02
∞	−0.0125	0.408	1.024	1.436	1.686	1.979	2.094	2.255

TABLE 1. Values of J for several positive values of ϵ

a good approximation to the inertial migration velocity. For that reason, figure 7 compares the computed results for J to the results predicted by the Vasseur–Cox theory for $\epsilon = 0.2$.

Up to this point, both v_s and G have been assumed positive. If both v_s and G are negative, the results are the same as when both parameters are positive. Different behaviour is observed if either v_s or G is negative. The results for $v_s < 0$ and $G > 0$ are the same as the results for $v_s > 0$ and $G < 0$. Thus it is necessary to consider only the former case. This situation corresponds to negative values of ϵ. For negative values of ϵ, the sign of J is opposite to the sign of the lift force. Figure 8 shows J versus l_* for $\epsilon = -1$. For l_* greater than about 0.5, the lift force points towards the

l_*	$\epsilon=-0.2$	-0.4	-0.6	-0.8	-1.0	-1.5	-2.0	$-\infty$
0.1	−2.90	−1.46	−0.952	−0.695	−0.542	−0.338	−0.223	0.143
0.2	−2.55	−1.34	−0.844	−0.589	−0.435	−0.230	−0.114	0.255
0.4	−1.68	−0.980	−0.566	−0.334	−0.191	0.001	0.110	0.455
0.6	−1.11	−0.704	−0.340	−0.119	0.018	0.204	0.308	0.631
0.8	−0.745	−0.492	−0.157	0.057	0.191	0.371	0.471	0.771
1.0	−0.504	−0.317	0.0015	0.211	0.342	0.515	0.610	0.886
1.2	−0.358	−0.178	0.146	0.362	0.495	0.666	0.757	1.01
1.4	−0.239	−0.048	0.287	0.508	0.642	0.809	0.896	1.12
1.6	−0.162	0.051	0.404	0.634	0.771	0.937	1.02	1.22
1.8	−0.111	0.126	0.501	0.743	0.884	1.05	1.13	1.30
2.0	−0.076	0.182	0.576	0.827	0.972	1.14	1.22	1.37
3.0	−0.016	0.314	0.805	1.12	1.30	1.51	1.59	1.69
4.0	−0.003	0.354	0.898	1.25	1.46	1.70	1.80	1.89
5.0	−0.0007	0.370	0.939	1.32	1.54	1.81	1.91	2.02
∞	−0.0125	0.408	1.024	1.436	1.686	1.979	2.094	2.255

TABLE 2. Values of J for several negative values of ϵ

wall. However, at smaller separations, the lift force becomes repulsive. Such behaviour is to be expected on the basis of the Cox–Hsu result in (4.17).

The values of J are given in tables 1 and 2 for several values of ϵ. For values outside the range given in the tables, the asymptotic results of Saffman, Cox & Hsu, Vasseur & Cox, and this paper may be used to approximate the lift force. For small values of l_*, either the Cox–Hsu or Vasseur–Cox theories give useful approximations. For values of l_* larger than 5, the large-distance form in (4.14) may be used to estimate J^w for $\epsilon > 1$. For small values of ϵ, wall effects are small enough to be negligible for most purposes when $l_* > 5$. Thus, by combining the results in tables 1 and 2 with the various asymptotic limits, it should be possible to develop useful approximate fits to the lift forces.

5. Conclusions

The main result of this paper is the expression for the partial Fourier transform of the wall-induced component of the inertial migration velocity in (4.7). By performing the two-dimensional integral in (3.5), one may obtain the wall-induced component of the inertial migration velocity, v_m^w. When the distance from the wall is small compared to $(\nu/G)^{\frac{1}{2}}$, the result reduces to the Cox–Hsu result. For small values of ϵ, the results derived in this paper are consistent with the result derived by Vasseur & Cox (1977) for a sphere translating parallel to a flat wall in a stagnant fluid. Figures 4, 6, 7, and 8 show results predicted for J as defined in (4.15). When l is very large compared to $(\nu/G)^{\frac{1}{2}}$, a universal expression for the wall-induced lift applies. This result is given in (4.14). Tables 1 and 2 give values of the wall-induced lift over a range of ϵ for values of l that are of order $(\nu/G)^{\frac{1}{2}}$.

The results presented in this paper fill a gap between the results presented for wall-bounded flows by Cox & Hsu (1976) and Vasseur & Cox (1977) and the results presented for unbounded fluids by Saffman (1965) and McLaughlin (1991). Although the results of this paper are for a linear flow next to a single vertical wall, the method could be used to compute results for a plane Couette flow between two vertical walls. However, it appears to be very difficult to generalize the analysis of this paper to include parabolic flows. Using the Cox–Brenner theory, Cox & Hsu (1976) and

Vasseur & Cox (1976) derived results for parabolic flows as well as linear flows. The restriction on their theory is that the wall must lie within the 'inner' region of the disturbance flow created by the particle.

If both v_s and G are positive ($\epsilon > 0$), the lift force on the sphere points away from the wall except for cases in which ϵ is small and l_* is large. The situation corresponds to a negatively buoyant particle in an upward shear flow. The same behaviour occurs for a positively buoyant particle in a downward shear flow.

A positively buoyant particle in an upward shear flow ($v_s < 0$ and $G > 0$) or a negatively buoyant particle in a downward shear flow ($v_s > 0$ and $G < 0$) experiences a lift force that points towards the wall if it is at sufficiently large distances from the wall provided that the magnitude of ϵ is not too small. For negative values of ϵ that are large in magnitude, the lift force points towards the wall except for $l_* \ll 1/\epsilon$ (see (4.17).) However, for negative values of ϵ that are small in magnitude, the lift force points away from the wall for all values of l_*.

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REFERENCES

ANTOSIEWICZ, H. A. 1965 Bessel functions of fractional order. In *Handbook of Mathematical Functions* (ed. M. Abramowitz & I. Stegun), pp. 435–478. Dover.

BENDER, C. M. & ORSZAG, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill.

BRENNER, H. 1966 *Adv. Chem. Engng* **6**, 287–438.

COX, R. G. & BRENNER, H. 1968 *Chem. Engng Sci.* **23**, 147–173.

COX, R. G. & HSU, S. K. 1977 *Intl J. Multiphase Flow* **3**, 201–222.

COX, R. G. & MASON, S. G. 1971 *Ann. Rev. Fluid Mech.* **3**, 291–316.

DAVIS, P. J. 1965 Gamma function and related functions. In *Handbook of Mathematical Functions* (ed. M. Abramowitz & I. Stegun), pp. 253–293. Dover.

DREW, D. A. 1988 *Chem. Engng Sci.* **43**, 769–773.

DREW, D. A., SCHONBERG, J. A. & BELFORT, G. 1991 *Chem. Engng Sci.* **46**, 3219–3224.

GOLDSMITH, H. L. & MASON, S. G. 1967 The microrheology of dispersions. In *Rheology, Theory and Applications* (ed. F. R. Eirich), vol. 4, pp. 85–250. Academic.

LEAL, L. G. 1980 *Ann. Rev. Fluid Mech.* **12**, 435–476.

MCLAUGHLIN, J. B. 1991 *J. Fluid Mech.* **224**, 261–274.

SAFFMAN, P. G. 1965 *J. Fluid Mech.* **22**, 385–400 (and Corrigendum **31**, 1968, 624).

SCHONBERG, J. A. & HINCH, E. J. 1989 *J. Fluid Mech.* **203**, 517–524.

SHIBATA, M. & MEI, C. C. 1990 *Phys. Fluids* A **2**, 1094–1104.

VASSEUR, P. & COX, R. G. 1976 *J. Fluid Mech.* **78**, 385–413.

VASSEUR, P. & COX, R. G. 1977 *J. Fluid Mech.* **80**, 561–591.